

# ANALYSIS OF MULTIPATCH DISCONTINUOUS GALERKIN IGA APPROXIMATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS \*

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**Abstract.** In this work, we study the approximation properties of multi-patch dG-IgA methods, that apply the multipatch Isogeometric Analysis (IgA) discretization concept and the discontinuous Galerkin (dG) technique on the interfaces between the patches, for solving linear diffusion problems with diffusion coefficients that may be discontinuous across the patch interfaces. The computational domain is divided into non-overlapping sub-domains, called patches in IgA, where  $B$ -splines, or NURBS finite dimensional approximations spaces are constructed. The solution of the problem is approximated in every sub-domain without imposing any matching grid conditions and without any continuity requirements for the discrete solution across the interfaces. Numerical fluxes with interior penalty jump terms are applied in order to treat the discontinuities of the discrete solution on the interfaces. We provide a rigorous a priori discretization error analysis for problems set in 2d- and 3d- dimensional domains, with solutions belonging to  $W^{l,p}$ ,  $l \geq 2$ ,  $p \in (2d/(d+2(l-1)), 2]$ . In any case, we show optimal convergence rates of the discretization with respect to the dG - norm.

**Key words.** linear elliptic problems, discontinuous coefficients, discontinuous Galerkin discretization, Isogeometric Analysis, non-matching grids, low regularity solutions, a priori discretization error estimates

**AMS subject classifications.** 65N12, 65N15, 65N35

**1. Introduction.** The finite element methods (FEM) and, in particular, discontinuous Galerkin (dG) finite element methods are very often used for solving elliptic boundary value problems which arise from engineering applications, see, e.g., [19], [26]. Although the isoparametric FEM and even FEM with curved finite elements have been proposed and analyzed long time ago, cf. [35], [7], [19], the quality of the numerical results for realistic problems in complicated geometries depends on the quality of the discretized geometry (triangulation of the domain), which is usually performed by a mesh generator. In many practical situations, extremely fine meshes are required around fine-scale geometrical objects, singular corner points etc. in order to achieve numerical solutions with desired resolution. This fact leads to an increased number of degrees of freedom, and thus to an increased overall computational cost for solving the discrete problem, see, e.g., [33] for fluid dynamics applications.

Recently, the Isogeometric Analysis (IgA) concept has been applied for approximating solutions of elliptic problems [20], [4]. IgA generalizes and improves the classical FE (even isoparametric FE) methodology in the following direction: complex technical computational domains can be exactly represented as images of some parameter domain, where the mappings are constructed by using superior classes of basis functions like  $B$ -Splines, or Non-Uniform Rational  $B$ -Splines (NURBS), see, e.g., [32] and [29]. The same class of functions is used to approximate the exact solution without increasing the computational cost for the computation of the resulting stiffness matrices [8], systematic  $hpk$  refinement procedures can easily be developed [9], and, last but not least, the method can be materialized in parallel environment incorporating fast domain decomposition solvers [23], [10], [2].

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During the last two decades, there has been an increasing interest in discontinuous Galerkin finite element methods for the numerical solution of several types of partial differential equations, see, e.g., [26]. This is due to the advantages of the local approximation spaces without continuity requirements that dG methods offer, see, e.g., [3], [27], [30] and [12].

In this paper, we combine the best features of the two aforementioned methods, and develop a powerful discretization method that we call multipatch discontinuous Galerkin Isogeometric Analysis (dG-IgA). In particular, we study and analyze the dG-IgA approximation properties to elliptic boundary value problems with discontinuous coefficients. It is well known that the solutions of this type of problems are in general not enough smooth, see, e.g. [21], [24], and the approximate method can not produce an (optimal) accurate solution. The problem is set in a complex, bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , which is subdivided in a union of non-overlapping sub-domains, say  $\mathcal{S}(\Omega) := \{\Omega_i\}_{i=1}^N$ . Let us assume that the discontinuity of the diffusion coefficients is only observed across sub-domain boundaries (interfaces). The weak solution of the problem is approximated in every sub-domain applying IgA methodology, [4], without matching grid conditions along the interfaces, as well without imposing continuity requirements for the approximation spaces across the interfaces. By construction, dG methods use discontinuous approximation spaces utilizing numerical fluxes on the interfaces, [22], and have been efficiently used for solving problems on non-matching grids in the past, [12], [13], [16]. Here, emulating the dG finite element methods, the numerical scheme is formulated by applying numerical fluxes with interior penalty coefficients on the interfaces of the sub-domains (patches), and using IgA formulations in every patch independently. A crucial point in the presented work, is the expression of the numerical flux interface terms as a sum over the micro-elements edges taking note of the non-matching sub-domain grids. This gives the opportunity to proceed in the error analysis by applying the trace inequalities locally as in dG finite element methods. There are many papers, which present dG finite element approximations for elliptic problems, see, e.g., [3], [31], the monographs [30], [27], and, in particular, for the discontinuous coefficient case, [12], [28]. However, there are only a few publications on the dG-IgA and their analysis. In [6], the author presented discretization error estimates for the dG-IgA of plane (2d) diffusion problems on meshes matching across the patch boundaries and under the assumption of sufficiently smooth solutions. This analysis obviously carries over to plane linear elasticity problems which have recently been studied numerically in [2]. In [14], the dG technology has been used to handle no-slip boundary conditions and multi-patch geometries for IgA of Darcy-Stokes-Brinkman equations. DG-IgA discretizations of heterogeneous diffusion problems on open and closed surfaces, which are given by a multipatch NURBS representation, are constructed and rigorously analyzed in [25].

In the first part of this paper, we give a priori error estimates in the  $\|\cdot\|_{dG}$  norm under the usual regularity assumption imposed on the exact solution, i.e.  $u \in W^{1,2}(\Omega) \cap W^{l \geq 2, 2}(\mathcal{S}(\Omega))$ . Next, we consider the model problem with low regularity solution  $u \in W^{1,2}(\Omega) \cap W^{l,p}(\mathcal{S}(\Omega))$ , with  $l \geq 2$  and  $p \in (\frac{2d}{d+2(l-1)}, 2)$ , and derive error estimates in the  $\|\cdot\|_{dG}$ . These estimates are optimal with respect to the space size discretization. We note that the error analysis in the case of low regularity solutions includes many ingredients of the dG FE error analysis of [34] and [28]. To the best of our knowledge, optimal error analysis for IgA discretizations combined with dG techniques for solving elliptic problems with discontinuous coefficients in general domains  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , have not been yet presented in the literature.

The paper is organized as follows. In Section 2, our model diffusion problem is described. Section 3 introduces some notations. The local  $\mathbb{B}_h(\mathcal{S}(\Omega))$  approximation space and the numerical scheme are also presented. Several auxiliary results and the analysis of the method for the case of usual regularity solutions are provided in Section 4. Section 5 is devoted to the analysis of the method for low regularity solutions. Section 6 includes several numerical examples that verify the theoretical convergence rates. Finally, we draw some conclusions.

**2. The model problem.** Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with the boundary  $\partial\Omega$ . For simplicity, we restrict our study to the model problem

$$(2.1) \quad -\operatorname{div}(\alpha \nabla u) = f \text{ in } \Omega, \text{ and } u = u_D \text{ on } \partial\Omega,$$

where  $f$  and  $u_D$  are given smooth data. In (2.1),  $\alpha$  is the diffusion coefficient and assume be bounded by above and below by strictly positive constants.

The weak formulation is to find a function  $u \in W^{1,2}(\Omega)$  such that  $u := u_D$  on  $\partial\Omega$  and satisfies

$$(2.2a) \quad a(u, \phi) = l(\phi), \quad \forall \phi \in W_0^{1,2}(\Omega),$$

where

$$(2.2b) \quad a(u, \phi) = \int_{\Omega} \alpha \nabla u \nabla \phi \, dx, \quad \text{and} \quad l(\phi) = \int_{\Omega} f \phi \, dx.$$

Results concerning the existence and uniqueness of the solution  $u$  of problem (2.2) can be derived by a simple application of Lax-Milgram Lemma, [15]. To avoid unnecessary long formulas below, we only considered in (2.1) non-homogeneous Dirichlet boundary conditions on  $\partial\Omega$ . However, the analysis can be easily generalized to Neumann and Robin type boundary conditions on a part of  $\partial\Omega$ , since they are naturally introduced in the dG formulation.

**3. Preliminaries - dG notation.** Throughout this work, we denote by  $L^p(\Omega)$ ,  $p > 1$  the Lebesgue spaces for which  $\int_{\Omega} |u(x)|^p \, dx < \infty$ , endowed with the norm  $\|u\|_{L^p(\Omega)} = (\int_{\Omega} |u(x)|^p \, dx)^{\frac{1}{p}}$ . By  $\mathcal{D}(\Omega)$ , we define the the space of  $C^\infty$  functions with compact support in  $\Omega$ , and by  $C^k(\Omega)$  the set of functions with  $k$ -th order continues derivatives. In dealing with differential operators in Sobolev spaces, we use the following common conventions. For any (multi-index)  $\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha_j \geq 0, j = 1, \dots, d$ , with degree  $|\alpha| = \sum_{j=1}^d \alpha_j$ , we define the *differential operator*

$$(3.1) \quad D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \text{ with } D_j = \frac{\partial}{\partial x_j}, D^{(0, \dots, 0)} u = u.$$

We also denote by  $W^{l,p}(\Omega)$ ,  $l$  positive integer and  $1 \leq p \leq \infty$ , the Sobolev space functions endowed with the norm

$$(3.2a) \quad \|u\|_{W^{l,p}(\Omega)} = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

$$(3.2b) \quad \|u\|_{W^{l,\infty}(\Omega)} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{\infty}.$$

For more details for the above definitions, we refer [1]. In the sequel we write  $a \sim b$  if  $ca \leq b \leq Ca$ , where  $c, C$  are positive constants indpented of the mesh size.

In order to apply the IgA methodology for the problem (2.1), the domain  $\Omega$  is subdivided into a union of sub-domains  $\mathcal{S}(\Omega) := \{\Omega_i\}_{i=1}^N$ , such that

$$(3.3) \quad \bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i, \quad \text{with} \quad \Omega_i \cap \Omega_j = \emptyset, \text{ if } j \neq i.$$

The subdivision of  $\Omega$  assumed to be compatible with the discontinuities of  $\alpha$ , [12], [28]. In other words, the diffusion coefficient assumed to be constant in the interior of  $\Omega_i$  and its discontinuities can appear only on the interfaces  $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ .

As it is common in the IgA analysis, we assume a parametric domain  $\hat{D}$  of unit length, (e.g.  $\hat{D} = [0, 1]^d$ ). For any  $\Omega_i$ , we associate  $n = 1, \dots, d$  knot vectors  $\Xi_n^{(i)}$  on  $\hat{D}$ , which create a mesh  $T_{h_i, \hat{D}}^{(i)} = \{\hat{E}_m\}_{m=1}^{M_i}$ , where  $\hat{E}_m$  are the micro-elements, see details in [8]. We shall refer  $T_{h_i, \hat{D}}^{(i)}$  as the *parametric mesh* of  $\Omega_i$ . For every  $\hat{E}_m \in T_{h_i, \hat{D}}^{(i)}$  we denote by  $h_{\hat{E}_m}$  its diameter and by  $h_i = \max\{h_{\hat{E}_m}\}$  the meshsize of  $T_{h_i, \hat{D}}^{(i)}$ . We assume the following properties for every  $T_{h_i, \hat{D}}^{(i)}$ ,

- quasi-uniformity: for every  $\hat{E}_m \in T_{h_i, \hat{D}}^{(i)}$  holds  $h_i \sim h_{\hat{E}_m}$ ,
- for the micro-element edges  $e_{\hat{E}_m} \subset \partial\hat{E}_m$  holds  $h_{\hat{E}_m} \sim e_{\hat{E}_m}$ .

On every  $T_{h_i, \hat{D}}^{(i)}$ , we construct the finite dimensional space  $\hat{\mathbb{B}}_{h_i}^{(i)}$  spanned by  $\mathbb{B}$ -Spline basis functions of degree  $k$ , [8], [32],

$$(3.4a) \quad \hat{\mathbb{B}}_{h_i}^{(i)} = \text{span}\{\hat{B}_j^{(i)}(\hat{x})\}_{j=0}^{\dim(\hat{\mathbb{B}}_{h_i}^{(i)})},$$

where every  $\hat{B}_j^{(i)}(\hat{x})$  base function in (3.4a) is derived by means of tensor products of one-dimensional  $\mathbb{B}$ -Spline basis functions, e.g.

$$(3.4b) \quad \hat{B}_j^{(i)}(\hat{x}) = \hat{B}_{j_1}^{(i)}(\hat{x}_1) \cdots \hat{B}_{j_d}^{(i)}(\hat{x}_d).$$

For simplicity, we assume that the basis functions of every  $\hat{\mathbb{B}}_{h_i}^{(i)}, i = 1, \dots, N$  are of the same degree  $k$ . We denote by  $\tilde{D}_{\hat{E}}^{(i)}$  the support extension of  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$ .

Every sub-domain  $\Omega_i \in \mathcal{S}(\Omega), i = 1, \dots, N$ , is exactly represented through a parametrization (one-to-one mapping), [8], having the form

$$(3.5a) \quad \Phi_i : \hat{D} \rightarrow \Omega_i, \quad \Phi_i(\hat{x}) = \sum_j C_j^{(i)} \hat{B}_j^{(i)}(\hat{x}) := x \in \Omega_i,$$

$$(3.5b) \quad \text{with} \quad \hat{x} = \Psi_i(x) := \Phi_i^{-1}(x),$$

where  $C_j^{(i)}$  are the control points. Using  $\Phi_i$ , we construct a mesh  $T_{h_i, \Omega_i}^{(i)} = \{E_m\}_{m=1}^{M_i}$  for every  $\Omega_i$ , whose vertices are the images of the vertices of the corresponding mesh  $T_{h_i, \hat{D}}^{(i)}$  through  $\Phi_i$ . If  $h_{\Omega_i} = \max\{h_{E_m}\}$ ,  $E_m \in T_{h_i, \Omega_i}^{(i)}$  is the sub-domain  $\Omega_i$  mesh size, then based on definition (3.5) of  $\Phi_i$ , there is a constant  $C := C(\|\Phi_i\|_\infty)$  such that  $h_i \sim Ch_{\Omega_i}$ . In what follows, we denote the sub-domain mesh size by  $h_i$  without the constant  $C := C(\|\Phi_i\|_\infty)$  explicitly appearing.

The mesh of  $\Omega$  is considered to be  $T_h(\Omega) = \bigcup_{i=1}^N T_{h_i, \Omega_i}^{(i)}$ , where we note that there are no matching mesh requirements on the interior interfaces  $F_{ij} = \partial\Omega_i \cap \partial\Omega_j, i \neq j$ .

For the sake of brevity in our notations, the interior faces of the boundary of the sub-domains are denoted by  $\mathcal{F}_I$  and the collection of the faces that belong to  $\partial\Omega$  by  $\mathcal{F}_B$ , e.g.  $F \in \mathcal{F}_B$  if there is a  $\Omega_i$  such that  $F = \partial\Omega_i \cap \partial\Omega$ . We denote the set of all sub-domain faces by  $\mathcal{F}$ .

Lastly, we define on  $\Omega$  the finite dimensional  $\mathbb{B}$ -Spline space  $\mathbb{B}_h(\mathcal{S}(\Omega)) = \mathbb{B}_{h_1}^{(1)} \times \dots \times \mathbb{B}_{h_N}^{(N)}$ , where every  $\mathbb{B}_{h_i}^{(i)}$  is defined on  $T_{h_i, \Omega_i}^{(i)}$  as follows

$$(3.6) \quad \mathbb{B}_{h_i}^{(i)} := \{B_{h_i}^{(i)}|_{\Omega_i} : B_h^{(i)}(\hat{x}) = \hat{B}_h^{(i)} \circ \Psi_i(x), \forall \hat{B}_h^{(i)} \in \hat{\mathbb{B}}_{h_i}^{(i)}\}.$$

We define the union support in physical sub-domain  $\Omega_i$  as  $D_E^{(i)} := \Phi(\tilde{D}_E^{(i)})$ . Since  $\Phi_i(\hat{x}) \in \hat{\mathbb{B}}_h^{(i)}$ , the components  $\Phi_{1,i}, \dots, \Phi_{d,i} \in \hat{\mathbb{B}}_h^{(i)}$  are smooth functions and hence there exist constants  $c_m, c_M$  such that

$$(3.7) \quad c_m \leq |\det(\Phi_i'(\hat{x}))| \leq c_M, \quad i = 1, \dots, d, \quad \text{for all } \hat{x} \in \hat{D}$$

where  $\Phi_i'(\hat{x})$  denotes the Jacobian matrix  $\frac{\partial(x_1, \dots, x_d)}{\partial(\hat{x}_1, \dots, \hat{x}_d)}$ .

Now, for any  $\hat{u} \in W^{m,p}(\hat{D})$ ,  $m \geq 0, p > 1$ , we define the function

$$(3.8) \quad \mathcal{U}(x) = \hat{u}(\Psi_i(x)), \quad x \in \Omega_i,$$

where  $\Psi$  is defined in (3.5b). For the error analysis presented below, it is necessary to show the relation

$$(3.9) \quad C_m \|\hat{u}\|_{W^{m,p}(\hat{D})} \leq \|\mathcal{U}\|_{W^{m,p}(\Omega_i)} \leq C_M \|\hat{u}\|_{W^{m,p}(\hat{D})},$$

where the constants  $C_m, C_M$  depending on

$$C_m := C_m(\max_{m_0 \leq m} (\|D^{m_0} \Phi_i\|_\infty), \|\det(\Psi_i')\|_\infty)$$

and

$$C_M := C_M(\max_{m_0 \leq m} (\|D^{m_0} \Psi_i\|_\infty), \|\det(\Phi_i')\|_\infty)$$

correspondingly.

Indeed, for any  $\hat{u} \in W^{m,p}(\hat{D})$  we can find a sequence  $\{\hat{u}_j\} \subset C^\infty(\bar{\hat{D}})$  converging to  $\hat{u}$  in  $\|\cdot\|_{W^{m,p}(\hat{D})}$ , by the chain rule in (3.8) we obtain

$$(3.10) \quad D_x(\Psi_i(x))^{-1} D\mathcal{U}_j(x) = D\hat{u}_j(\Psi_i(x)).$$

Then for any multi-index  $m$  we can get the following formula

$$(3.11) \quad D^m \mathcal{U}_j(x) = \sum_{m_0 \leq m} P_{m, m_0}(x) D^{m_0} \mathcal{U}_j(x),$$

where  $P_{m, m_0}(x) \in \mathbb{B}_h^{(i)}$  is a polynomial of degree less than  $k$  and includes the various derivatives of  $\Psi_i(x)$ . Multiplying (3.11) by  $\varphi(x) \in \mathcal{D}(\Omega_i)$ , and integrating by parts we have

$$(3.12) \quad (-1)^{|m|} \int_{\Omega_i} \mathcal{U}_j(x) D^m \varphi(x) dx = \sum_{m_0 \leq m} \int_{\Omega_i} P_{m, m_0}(x) D^{m_0} \mathcal{U}_j(x) \varphi(x) dx.$$

We transfer the integral in (3.12) to integrals over  $\hat{D}$  and use the change of variable  $x = \Phi_i(\hat{x})$  to obtain

$$(3.13) \quad (-1)^{|m|} \int_{\hat{D}} \hat{u}_j(\hat{x}) D^m \varphi(\Phi_i(\hat{x})) |det(\Phi_i'(\hat{x}))| d\hat{x} = \sum_{m_0 \leq m} \int_{\hat{D}} P_{m, m_0}(\Phi_i(\hat{x})) D^{m_0} \hat{u}_j(\hat{x}) \varphi(\Phi_i(\hat{x})) |det(\Phi_i'(\hat{x}))| d\hat{x}.$$

But it holds that  $D^{m_0} \hat{u}_j \rightarrow D^{m_0} \hat{u}$  in  $\|\cdot\|_{L^p(\hat{D})}$ , thus taking the limit  $j \rightarrow \infty$  in (3.13) and transferring the integrals back to  $\Omega_i$ , we can derive (3.12) with respect to  $\mathcal{U}$ . We conclude that (3.11) holds in the distributional sense, and therefore

$$(3.14) \quad \int_{\Omega_i} |D^m \mathcal{U}(x)|^p dx \leq C_p \int_{\Omega_i} \sum_{m_0 \leq m} |P_{m, m_0}(x) D^{m_0} \mathcal{U}(x)|^p dx \leq C_p \max_{m_0 \leq m} \left( \max_{x \in \Omega_i} (P_{m, m_0}(x)) \right) \sum_{m_0 \leq m} \int_{\Omega_i} |D^{m_0} \mathcal{U}(x)|^p dx \leq C_p \max_{m_0 \leq m} \left( \max_{x \in \Omega_i} (P_{m, m_0}(x)) \right) \max_{\hat{x} \in \hat{D}} (|det(\Phi_i'(\hat{x}))|) \sum_{m_0 \leq m} \int_{\hat{D}} |D^{m_0} \hat{u}(\hat{x})|^p d\hat{x} \leq C \left( \max_{m_0 \leq m} (\|D^{m_0} \Psi_i(x)\|_\infty, \|det(\Phi_i'(\hat{x}))\|_\infty) \right) \sum_{m_0 \leq m} |D^{m_0} \hat{u}(\hat{x})|_{W^{m_0, p}(\hat{D})}^p.$$

This proves the “right inequality” of (3.9). The “left inequality” of (3.9) can be derived following the same arguments as above using the change of variable  $\hat{x} = \Psi_i(x)$ .

**3.1. The numerical scheme.** We use the  $\mathbb{B}$ -Spline spaces  $\mathbb{B}_h^{(i)}$  defined in (3.6) for approximating the solution of (2.2) in every sub-domain  $\Omega_i$ . Continuity requirements for  $\mathbb{B}_h(\mathcal{S}(\Omega))$  are not imposed on the interfaces  $F_{ij}$  of the sub-domains, clearly  $\mathbb{B}_h(\mathcal{S}(\Omega)) \subset L^2(\Omega)$  but  $\mathbb{B}_h(\mathcal{S}(\Omega)) \not\subset W^{1,2}(\Omega)$ . Thus, the problem (2.2) is discretized by discontinuous Galerkin techniques on  $F_{ij}$ , [12]. Using the notation  $\phi_h^{(i)} := \phi_h|_{\Omega_i}$ , we define the average and the jump of  $\phi_h$  on  $F_{ij} \in \mathcal{F}_I$  respectively by

$$(3.15a) \quad \{\phi_h\} := \frac{1}{2}(\phi_h^{(i)} + \phi_h^{(j)}), \quad \llbracket \phi_h \rrbracket := \phi_h^{(i)} - \phi_h^{(j)},$$

and for  $F_i \in \mathcal{F}_B$

$$(3.15b) \quad \{\phi_h\} := \phi_h^{(i)}, \quad \llbracket \phi_h \rrbracket := \phi_h^{(i)}.$$

The dG-IgA method reads as follows: find  $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$  such that

$$(3.16a) \quad a_h(u_h, \phi_h) = l(\phi_h) + p_D(u_D, \phi_h), \quad \forall \phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega)),$$

where

$$(3.16b) \quad a_h(u_h, \phi_h) = \sum_{i=1}^N a_i(u_h, \phi_h) - \sum_{F_{ij} \in \mathcal{F}} \frac{1}{2} s_i(u_h, \phi_h) + p_i(u_h, \phi_h),$$

with the bi-linear forms

$$(3.16c) \quad a_i(u_h, \phi_h) = \int_{\Omega_i} \alpha \nabla u_h \nabla \phi_h \, dx,$$

$$(3.16d) \quad s_i(u_h, \phi_h) = \int_{F_{ij} \in \mathcal{F}} \{\alpha \nabla u_h\} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket \, ds,$$

$$(3.16e) \quad p_i(u_h, \phi_h) = \begin{cases} p_{i_I}(u_h, \phi_h) & = \int_{F_{ij} \in \mathcal{F}_I} \left( \frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i} \right) \llbracket u_h \rrbracket \llbracket \phi_h \rrbracket \, ds, \\ p_{i_B}(u_h, \phi_h) & = \int_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \llbracket u_h \rrbracket \llbracket \phi_h \rrbracket \, ds, \end{cases}$$

$$(3.16f) \quad p_D(u_D, \phi_h) = \int_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} u_D \phi_h \, ds,$$

where the unit normal vector  $\mathbf{n}_{F_{ij}}$  is oriented from  $\Omega_i$  towards the interior of  $\Omega_j$  and the parameter  $\mu > 0$  will be specified later in the error analysis, cf. [12].

For notation convenience in what follows, we will use the same expression

$$\int_{F_{ij} \in \mathcal{F}} \left( \frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i} \right) \llbracket u_h \rrbracket \llbracket \phi_h \rrbracket \, ds,$$

for both cases,  $F_{ij} \in \mathcal{F}_I$  and  $F_i \in \mathcal{F}_B$ . In the later case we will assume that  $\alpha^{(j)} = 0$ .

**REMARK 3.1.** We mention that, in [12], Symmetric Interior Penalty (SIP) dG formulations have been considered by introducing harmonic averages of the diffusion coefficients on the interface symmetric fluxes. Furthermore, harmonic averages of the two different grid sizes have been used to penalize the jumps. The possibility of using other averages for constructing the diffusion terms in front of the consistency and penalty terms has been analyzed in many other works as well, see, e.g. [28] and [18]. For simplicity of the presentation, we provide a rigorous analysis of the Incomplete Interior Penalty (IIP) forms (3.16d) and (3.16e). However, our analysis can easily be carried over to SIP dG-IgA that is preferred in practice for symmetric and positive definite (spd) variational problems due to the fact that the resulting systems of algebraic equations are spd and, therefore, can be solved by means of some preconditioned conjugate gradient method.

**4. Auxiliary results.** In order to proceed to error analysis, several auxiliary results must be shown for  $u \in W^{l,p}(\mathcal{S}(\Omega))$  and  $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ . The general frame of the proofs consists of three steps: (i) the required relations are expressed-proved on a parent element  $D_p$ , see Fig. 1, (ii) the relations are “transformed” to  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$  using an affine-linear mapping and scaling arguments, (iii) by virtue of the mappings  $\Phi_i$  defined in (3.6) and relations (3.9), we express the results in every  $\Omega_i$ .

Let  $D_p$  be the parent element e.g.  $[-x_b, x_b]^d \subset \mathbb{R}^d$ , with diameter  $H_p$ , see Fig. 1.  $D_p$  is convex simply connected domain, thus for any  $x \in \partial D_p$ ,  $\exists x_0 \in D_p$  such that

$$(4.1) \quad (x - x_0) \cdot \mathbf{n}_{\partial D_p} \geq C_{H_p}.$$

**LEMMA 4.1.** For any  $u \in W^{l,p}(D_p)$ ,  $l \geq 1, p > 1$  there is a  $C := C_{H_p, d, p}$  such that the following trace inequality holds true

$$(4.2) \quad \int_{\partial D_p} |u(s)|^p \, ds \leq C \left( \int_{D_p} |\nabla u(x)|^p \, dx + \int_{D_p} |u(x)|^p \, dx \right).$$

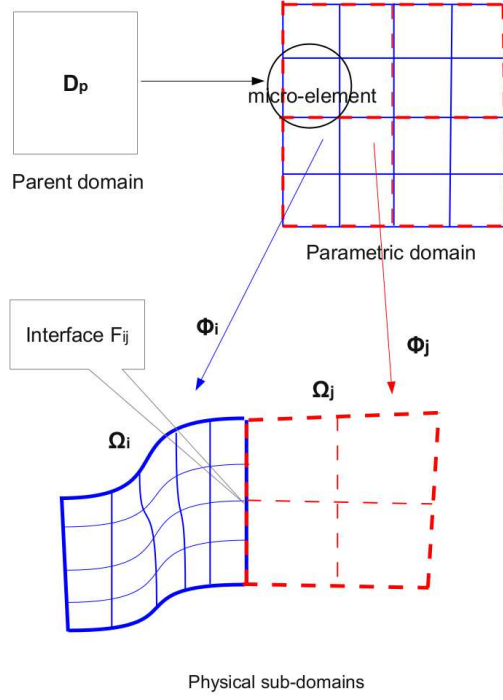


FIG. 1. The parent element, the parametric domain and two adjacent sub-domains.

*Proof.* For  $r = (x - x_0)$  we have

$$(4.3) \quad \int_{D_p} \nabla |u|^p \cdot r \, dx = \sum_{i=1}^d \int_{D_p} p |u|^{p-2} u \frac{\partial u}{\partial x_i} r_i \, dx = p \int_{D_p} |u|^{p-2} u \nabla u \cdot r \, dx.$$

The application of divergence theorem gives

$$(4.4) \quad \int_{D_p} \nabla |u|^p \cdot r \, dx = \int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p} \, ds - \int_{D_p} |u|^p \operatorname{div}(r) \, dx.$$

By (4.1), (4.3) and (4.4) it follows that

$$\int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p} \, ds = p \int_{D_p} |u|^{p-2} u \nabla u \cdot r \, dx + \int_{D_p} |u|^p \operatorname{div}(r) \, dx$$



and by (4.1), we get

$$(4.5) \quad C_{H_p} \int_{\partial D_p} |u|^p ds \leq p \int_{D_p} |u|^{p-2} u \nabla u \cdot r dx + \int_{D_p} |u|^p \operatorname{div}(r) dx.$$

Applying Hölder and Youngs inequalities, we have

$$\begin{aligned} \int_{\partial D_p} |u|^p ds &\leq C_{H_p} \left( C_{1,p} \left( \int_{D_p} |u|^p dx + |\nabla u|^p dx \right) + C_d \int_{D_p} |u|^p dx \right) \\ &\leq C_{H_p, d, p} \left( \int_{D_p} |u|^p dx + \int_{D_p} |\nabla u|^p dx \right) \\ &= C_{H_p, d, p} \left( \|u\|_{L^p(D_p)}^p + \|\nabla u\|_{L^p(D_p)}^p \right). \end{aligned}$$

□

We point out that similar proof has been given in [16] in case of  $p = 2$ .

$D_p$  can be considered as a reference element of any micro-element  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$  with the linear affine map

$$(4.6) \quad \phi_{\hat{E}} : D_p \rightarrow \hat{E} \in T_{h_i, \hat{D}}^{(i)}, \quad \phi_{\hat{E}}(x_{D_p}) = Bx_{D_p} + b,$$

where  $|\det(B)| = |\hat{E}|$ , see [5]. By (4.6), we have that  $|u|_{W^{l,p}(D_p)} = h_{\hat{E}}^{l-\frac{d}{p}} |\hat{u}|_{W^{l,p}(\hat{E})}$  and then we deduce by (4.2) that

$$h_{\hat{E}}^{-(d-1)} \int_{e \in \partial \hat{E}} |u|^p ds \leq C \left( h_{\hat{E}}^{(0-\frac{d}{p})p} \int_{\hat{E}} |u|^p dx + h_{\hat{E}}^{p(1-\frac{d}{p})} \int_{\hat{E}} |\nabla u|^p dx \right)$$

which directly gives

$$(4.7) \quad \int_{e \in \partial \hat{E}} |u|^p ds \leq C \left( \frac{1}{h_i} \int_{\hat{E}} |u|^p dx + h_i^{p-1} \int_{\hat{E}} |\nabla u|^p dx \right), \quad \forall \hat{E} \in T_{h_i, \hat{D}}^{(i)}.$$

Summing over all micro-elements  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$ , we have

$$(4.8) \quad \int_{\hat{F}_i \in \partial \hat{D}} |u|^p ds \leq C \left( \frac{1}{h_i} \int_{\hat{D}} |u|^p dx + h_i^{p-1} \int_{\hat{D}} |\nabla u|^p dx \right).$$

Finally, by making use of (3.9), we get the trace inequality expressed on every sub-domain

$$(4.9) \quad \int_{F_{ij} \in \mathcal{F}} |u|^p ds \leq C \left( \frac{1}{h_i} \int_{\Omega_i} |u|^p dx + h_i^{p-1} \int_{\Omega_i} |\nabla u|^p dx \right),$$

where the constant  $C$  is determined according to the  $C_m, C_M$  in (3.9).

LEMMA 4.2. (*Inverse estimates*) For all  $\phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}$  defined on  $T_{h_i, \hat{D}}^{(i)}$ , there is a constant  $C$  depended on mesh quasi-uniformity parameters of the mesh but not on  $h_i$ , such that

$$(4.10) \quad \|\nabla \phi_h\|_{L^p(\hat{D})}^p \leq \frac{C}{h_i^p} \|\phi_h\|_{L^p(\hat{D})}^p$$

*Proof.* The restriction of  $\phi_h|_{\hat{E}}$  is a  $B$ -Spline polynomial of the same order. Considering the same polynomial space on the  $D_p$  and by the equivalence of the norms on  $D_p$  we have, [5],

$$(4.11) \quad \|\nabla \phi_h\|_{L^p(D_p)}^p \leq C_{D_p} \|\phi_h\|_{L^p(D_p)}^p.$$

Applying scaling arguments and the mesh quasi-uniformity properties of  $T_{h_i, \hat{D}}^{(i)}$ , the left and the right hand side of (4.11) can be expressed on every  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$  as

$$(4.12) \quad h_i^{p-\frac{d}{p}p} \|\nabla \phi_h\|_{L^p(\hat{E})}^p \leq C h_i^{-\frac{d}{p}p} \|\phi_h\|_{L^p(\hat{E})}^p,$$

summing over all in (4.12)  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$ , we can easily deduce (4.10).

□

LEMMA 4.3. (*trace inequality for  $\phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}$* ) For all  $\phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}$  defined on  $T_{h_i, \hat{D}}^{(i)}$  and for all  $\hat{F}_i \in \partial \hat{D}$ , there is a constant  $C$  depended on mesh quasi-uniformity parameters of the mesh but not on  $h_i$ , such that

$$(4.13) \quad \|\phi_h\|_{L^p(\hat{F}_i \in \partial \hat{D})}^p \leq \frac{C}{h_i^p} \|\phi_h\|_{L^p(\hat{D})}^p$$

*Proof.* Applying the same scaling arguments as before and using the local quasi-uniformity of  $T_{h_i, \hat{D}}^{(i)}$ , that is for every  $\hat{e} \in \partial \hat{E}$  holds  $|\hat{e}| \sim h_i$ , we can show the following *local trace inequality*

$$(4.14) \quad \|\phi_h\|_{L^p(\hat{e} \in \partial \hat{E})}^p \leq C h_i^{-p} \|\phi_h\|_{L^p(\hat{E})}^p$$

summing over all  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$  that have an edge on  $\hat{F}_i$  we deduce (4.13).

□

Next a Lemma for the relation among the  $|\phi_h|_{W^{l,p}(\hat{D})}$  and  $|\phi_h|_{W^{m,p}(\hat{D})}$ .

LEMMA 4.4. Let  $\phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}$  such that  $\phi_h \in W^{l,p}(\hat{E}) \cap W^{m,q}(\hat{E})$ ,  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$ , and  $0 \leq m \leq l$ ,  $1 \leq p, q \leq \infty$ . Then there is a constant  $C := C(l, p, m, q)$  depended on mesh quasi-uniformity parameters of the mesh but not on  $h_i$ , such that

$$(4.15) \quad |\phi_h|_{W^{l,p}(\hat{E})} \leq C h_i^{m-l-\frac{d}{q}+\frac{d}{p}} |\phi_h|_{W^{m,q}(\hat{E})}.$$

*Proof.* We mimic the analysis of Chp 4 in [5]. For any  $\phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}|_{D_p}$ , we have that

$$(4.16) \quad |\phi_h|_{W^{l,p}(D_p)} \leq C |\phi_h|_{W^{m,q}(D_p)}, \quad \phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}|_{D_p}.$$

Using the scaling arguments as in proof of (4.7),

$$h_{\hat{E}}^{l-\frac{d}{p}} |\phi_h|_{W^{l,p}(\hat{E})} \leq C h_{\hat{E}}^{m-\frac{d}{q}} |\phi_h|_{W^{m,q}(\hat{E})}$$

which directly implies

$$(4.17) \quad |\phi_h|_{W^{l,p}(\hat{E})} \leq Ch_i^{m-l-\frac{d}{q}+\frac{d}{p}} |\phi_h|_{W^{m,q}(\hat{E})}, \quad \phi_h \in \hat{\mathbb{B}}_{h_i}^{(i)}.$$

For the particular case of  $m = l = 0$  in (4.15), we have that

$$(4.18) \quad \|\phi_h\|_{L^p(\hat{E})} \leq Ch_i^{d(\frac{1}{p}-\frac{1}{q})} \|\phi_h\|_{L^q(\hat{E})}.$$

□

**4.1. Analysis of the dG-IgA discretization.** Next, we study the convergence estimates of the method (3.16) under the following regularity assumption.

ASSUMPTION 1. *We assume for the solution  $u$  that  $u \in W_S^{l,2} := W^{1,2}(\Omega) \cap W^{l,2}(\mathcal{S}(\Omega))$ ,  $l \geq 2$ .*

We consider the enlarged space  $W_h^{l,2} := W_S^{l,2} + \mathbb{B}_h(\mathcal{S}(\Omega))$ , equipped with the broken dG-norm

$$(4.19) \quad \|u\|_{dG}^2 = \sum_{i=1}^N \left( \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(\Omega_i)}^2 + p_i(u^{(i)}, u^{(i)}) \right), \quad u \in W_h^{l,2}.$$

For the error analysis is necessary to show the continuity and coercivity properties of the bilinear form  $a_h(.,.)$  of (3.16). Initially, we give a bound for the consistency terms.

LEMMA 4.5. *For  $(u, \phi_h) \in W_h^{l,2} \times \mathbb{B}_h(\mathcal{S}(\Omega))$ , there are  $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$  such that for every  $F_{ij} \in \mathcal{F}_I$*

$$(4.20) \quad |s_i| = \left| \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} (\phi_h^{(i)} - \phi_h^{(j)}) ds \right| \leq \\ C_{1,\varepsilon} \left( h_i \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + h_j \alpha^{(j)} \|\nabla u^{(j)}\|_{L^2(F_{ij})}^2 \right) + \\ \frac{1}{C_{2,\varepsilon}} \left( \frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j} \right) \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}^2.$$

*Proof.* Expanding the terms and applying Cauchy-Schwartz inequality yields

$$|s_i| \leq C \left| \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} (\phi_h^{(i)} - \phi_h^{(j)}) ds \right| \leq C \left( \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})} + \alpha^{(j)} \|\nabla u^{(j)}\|_{L^2(F_{ij})} \right) \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}.$$

Applying Young's inequality:

$$\alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})} \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})} \leq C_{1,\varepsilon} h_i \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + \frac{\alpha^{(i)}}{C_{2,\varepsilon} h_i} \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}^2$$

we obtain

$$\begin{aligned} |s_i| &\leq C_{1,\varepsilon} h_i \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + C_{1,\varepsilon} h_j \alpha^{(j)} \|\nabla u^{(j)}\|_{L^2(F_{ij})}^2 + \\ &\frac{\alpha^{(i)}}{C_{2,\varepsilon} h_i} \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}^2 + \frac{\alpha^{(j)}}{C_{2,\varepsilon} h_j} \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}^2 = \\ &C_{1,\varepsilon} \left( h_i \alpha^{(i)} \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + h_j \alpha^{(j)} \|\nabla u^{(j)}\|_{L^2(F_{ij})}^2 \right) + \\ &\frac{1}{C_{2,\varepsilon}} \left( \frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j} \right) \|\phi_h^{(i)} - \phi_h^{(j)}\|_{L^2(F_{ij})}^2. \end{aligned}$$

□

REMARK 4.1. In case where  $F_i \in \mathcal{F}_B$ , the corresponding bound can be derived by setting in (4.20)  $\alpha^{(j)} = 0$  and  $\phi_h^{(j)} = 0$ .

LEMMA 4.6. (Discrete Coercivity) Suppose  $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$  is the dG-IgA solution derived by (3.16). There exist a  $C > 0$  independent of  $\alpha$  and  $h_i$ , such that

$$(4.21) \quad a_h(u_h, u_h) \geq C \|u_h\|_{dG}^2, \quad u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$$

*Proof.* By (3.16a), we have that

$$\begin{aligned} a_h(u_h, u_h) &= \sum_{i=1}^N a_i(u_h, u_h) - s_i(u_h, u_h) + p_i(u_h, u_h) = \\ &\sum_{i=1}^N \alpha_i \|\nabla u_h\|_{L^2(\Omega_i)}^2 - \sum_{F_{ij} \in \mathcal{F}} \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u_h\} \cdot \mathbf{n}_{F_{ij}} \llbracket u_h \rrbracket ds + \\ (4.22) \quad &\sum_{F_{ij} \in \mathcal{F}} \mu \left( \frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j} \right) \|\llbracket u_h \rrbracket\|_{L^2(F_{ij})}^2. \end{aligned}$$

For the second term on the right hand side, Lemma 4.5 and the trace inequality (4.13)

expressed on  $F_{ij} \in \mathcal{F}$  yield the bound

$$(4.23) \quad - \sum_{F_{ij} \in \mathcal{F}} \frac{1}{2} \int_{F_{ij}} \{ \alpha \nabla u_h \} \cdot \mathbf{n}_{F_{ij}} \llbracket u_h \rrbracket ds \geq \\ - C_{1,\varepsilon} \sum_{i=1}^N \alpha_i \|\nabla u_h\|_{L^2(\Omega_i)}^2 - \sum_{F_{ij} \in \mathcal{F}} \frac{1}{C_{2,\varepsilon}} \left( \frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j} \right) \|\llbracket u_h \rrbracket\|_{L^2(F_{ij})}^2.$$

Inserting (4.23) into (4.22) and choosing  $C_{1,\varepsilon} < \frac{1}{2}$  and  $\mu > \frac{2}{C_{2,\varepsilon}}$  we obtain (4.21).

□

LEMMA 4.7. (*Boundedness*) *There are  $C_1, C_2 > 0$  independent of  $h_i$  such that for all  $(u, \phi_h) \in W_h^{l,2} \times \mathbb{B}_h(\mathcal{S}(\Omega))$*

$$(4.24) \quad a_h(u, \phi_h) \leq C_1 \left( \|u\|_{dG}^2 + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 \right) + C_2 \|\phi_h\|_{dG}^2.$$

*Proof.* We have by (3.16a) that

$$(4.25) \quad a_h(u, \phi_h) = \sum_{i=1}^N \int_{\Omega_i} \alpha \nabla u \nabla \phi_h dx + \sum_{F_{ij} \in \mathcal{F}} \frac{1}{2} \int_{F_{ij}} \{ \alpha \nabla u \} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket ds + \\ \sum_{F_{ij} \in \mathcal{F}} \int_{F_{ij}} \left( \frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i} \right) \llbracket u \rrbracket \llbracket \phi_h \rrbracket ds = T_1 + T_2 + T_3.$$

Applying Cauchy-Schwartz inequality and consequently Young's inequality on every term in (4.25) yield the bounds

$$T_1 \leq C_1 \|u\|_{dG}^2 + C_2 \|\phi_h\|_{dG}^2.$$

For the term  $T_2$ , owing to the Lemma 4.5, we have

$$T_2 \leq \sum_{F_{ij} \in \mathcal{F}} \left( C_1 \alpha^{(i)} h_i \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + C_2 \left( \frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right) \\ \leq C_1 \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i \|\nabla u^{(i)}\|_{L^2(F_{ij})}^2 + C_2 \|\phi_h\|_{dG}^2,$$

$$T_3 \leq \sum_{F_{ij} \in \mathcal{F}} \left( \frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i} \right) \left( C_1 \|\llbracket u \rrbracket\|_{L^2(F_{ij})}^2 + C_2 \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right) \leq C_1 \|u\|_{dG}^2 + C_2 \|\phi_h\|_{dG}^2.$$

Substituting the bounds of  $T_1, T_2, T_3$  into (4.25), we can derive (4.24). □

In Chp 12 in [32],  $B$ -Spline quasi-interpolants, say  $\Pi_h$ , are defined for  $u \in W^{l,p}$  functions. Next, we consider the same quasi-interpolant and give an estimate on how well  $\Pi_h u$  approximates functions  $u \in W^{l,2}(\Omega_i)$  in  $\|\cdot\|_{dG}$ -norm.

LEMMA 4.8. *Let  $m, l \geq 2$  be positive integers with  $0 \leq m \leq l \leq k+1$  and let  $E = \Phi_i(\hat{E}), \hat{E} \in T_{h_i, \hat{D}}^{(i)}$ . For  $u \in W^{l,2}(\Omega_i)$  there exist a quasi-interpolant  $\Pi_h u \in \mathbb{B}_h^{(i)}$  and a constant  $C_i := C_i(\max_{l_0 < l} \|D^{l_0} \Phi_i\|_{L^\infty(\Omega_i)}, \|u\|_{W^{l,2}(\Omega_i)})$  such that*

$$(4.26) \quad \sum_{E \in T_{h_i, \Omega_i}^{(i)}} |u - \Pi_h u|_{W^{m,2}(E)}^2 \leq C_i h_i^{2(l-m)} \|u\|_{W^{l,2}(\Omega_i)}^2.$$

Further, for any  $F_{ij} \in \mathcal{F}$  the following estimates are true

$$(4.27a) \quad h_i \alpha^{(i)} \|(\nabla u^{(i)} - \nabla \Pi_h u^{(i)}) \cdot \mathbf{n}_{F_{ij}}\|_{L^2(F_{ij})}^2 \leq C_i h_i^{2l-2},$$

$$(4.27b) \quad \left(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\right) \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 \leq C_i \left(\alpha^{(i)} h_i^{2l-2} + \frac{\alpha^{(j)} h_i^{2l-1}}{h_j}\right),$$

$$(4.27c) \quad \|u - \Pi_h u\|_{dG}^2 \leq \sum_{i=1}^N C_i (h_i^{2l-2} + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(j)} \frac{h_i}{h_j} h_i^{2l-2}).$$

*Proof.* The proof of (4.26) is included in Lemma 5.1 (see below) if we set  $p = 2$ , see also [4].

Applying the trace inequality (4.9) for  $u := u^{(i)} - \Pi_h u^{(i)}$  and consequently using the approximation estimate (4.26) the result (4.27a) easily follows.

To prove (4.27b), we apply again (4.9) and obtain

$$\begin{aligned} \left(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\right) \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 &\leq \\ C_i \left(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\right) &\left(\frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(\Omega_i)}^2 + h_i \|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|_{L^2(\Omega_i)}^2\right) \leq \\ C_i \left(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\right) h_i^{2l-1} &\leq C_i \left(\alpha^{(i)} h_i^{2l-2} + \frac{\alpha^{(j)} h_i^{2l-1}}{h_j}\right) \end{aligned}$$

Recalling the approximation result (4.26) and using (4.27b) we can deduce estimate (4.27c).

□

In order to proceed and to give an estimate for the error  $\|u - u_h\|_{dG}$ , we need to show that the weak solution satisfies the form (3.16a).

LEMMA 4.9. (*Consistency of the weak solution.*) Under the Assumption 1, the weak solution  $u$  of the variational formulation (2.2) satisfies the dG-IgA variational identity (3.16), that is for all  $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ , we have

$$\begin{aligned} (4.28) \quad \sum_{i=1}^N \int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \sum_{F_{ij} \in \mathcal{F}_I} \left( \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds + \right. \\ \left. \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \int_{F_{ij}} [\![u]\!] [\![\phi_h]\!] \, ds \right) + \\ \sum_{F_i \in \mathcal{F}_B} \left( \int_{F_i} \alpha^{(i)} \nabla u \cdot \mathbf{n}_{F_i} \phi_h \, ds + \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u \phi_h \, ds \right) = \\ \sum_{i=1}^N \int_{\Omega_i} f \phi_h \, dx + \sum_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u_D \phi_h \, ds. \end{aligned}$$

*Proof.* We multiply (2.1) by  $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$  and integrating by parts on each sub-domain  $\Omega_i$  we get

$$\int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \int_{\partial \Omega_i} \alpha \nabla u \cdot \mathbf{n}_{\partial \Omega_i} \phi_h \, ds = \int_{\Omega_i} f \phi_h \, dx.$$

Summing over all sub-domains

$$(4.29) \quad \sum_{i=1}^N \int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \sum_{F_{ij} \in \mathcal{F}} \int_{F_{ij}} \llbracket \alpha \nabla u \phi_h \rrbracket \cdot \mathbf{n}_{F_{ij}} \, ds = \sum_{i=1}^N \int_{\Omega_i} f \phi_h \, dx.$$

The regularity Assumption 1 implies that  $\llbracket \alpha \nabla u \rrbracket \cdot \mathbf{n}_{F_{ij}} = 0$ . Making use of the identity

$$\llbracket ab \rrbracket = a_1 b_1 - a_2 b_2 = \{a\} \llbracket b \rrbracket + \llbracket a \rrbracket \{b\},$$

the relation (4.29) can be reformulated as

$$(4.30) \quad \sum_{i=1}^N \int_{\Omega_i} \alpha \nabla u \cdot \nabla \phi_h \, dx - \sum_{F_{ij} \in \mathcal{F}_I} \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket \, ds + \\ \sum_{F_i \in \mathcal{F}_B} \int_{F_i} \alpha \nabla u \cdot \mathbf{n}_{F_i} \phi_h \, ds = \sum_{i=1}^N \int_{\Omega_i} f \phi_h \, dx.$$

The continuity of  $u$  implies further that

$$(4.31) \quad \sum_{F_{ij} \in \mathcal{F}_I} \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \int_{F_{ij}} \llbracket u \rrbracket \llbracket \phi_h \rrbracket \, ds + \sum_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u \phi_h \, ds = \\ \sum_{F_i \in \mathcal{F}_B} \frac{\mu \alpha^{(i)}}{h_i} \int_{F_i} u_D \phi_h \, ds.$$

Adding (4.31) and (4.30) we obtain (4.28).

□

We can now give an error estimate in  $\|\cdot\|_{dG}$ -norm.

**THEOREM 4.10.** *Let  $u \in W_S^{l,2}$  solves (2.2) and let  $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$  solves the discrete problem (3.16). Then the error  $u - u_h$  satisfies*

$$(4.32) \quad \|u - u_h\|_{dG}^2 < \sum_{i=1}^N C_i \left( h_i^{2l-2} + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(j)} \frac{h_i}{h_j} h_i^{2l-2} \right),$$

where  $C_i := C(\max_{l_0 < l} \|D^{l_0} \Phi_i\|_{L^\infty(\Omega_i)}^l, \|u\|_{W^{l,2}(\Omega_i)})$ .

*Proof.*

Let  $\Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$  as in Lemma 4.8, by subtracting (4.28) from (3.16a) we get

$$a_h(u_h, \phi_h) = a_h(u, \phi_h),$$

and adding  $-a_h(\Pi_h u, \phi_h)$  on both sides

$$(4.33) \quad a_h(u_h - \Pi_h u, \phi_h) = a_h(u - \Pi_h u, \phi_h).$$

Note that  $u_h - \Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$ . Therefore we may set  $\phi_h = u_h - \Pi_h u$  in (4.33), and consequently applying Lemma 4.6 and Lemma 4.7 we find

$$(4.34) \quad \|u_h - \Pi_h u\|_{dG}^2 \leq C \left( \|u - \Pi_h u\|_{dG}^2 + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i \|\nabla(u^{(i)} - \Pi_h u^{(i)})\|_{L^2(F_{ij})}^2 \right)$$

Using the triangle inequality

$$(4.35) \quad \|u - u_h\|_{dG}^2 \leq \|u_h - \Pi_h u\|_{dG}^2 + \|u - \Pi_h u\|_{dG}^2$$

in (4.34) and consequently applying the estimates of (4.27) we can obtain (4.32).

□

**5. Low-Regularity solutions.** In this section, we investigate the convergence of the  $u_h$  produced by the dG-IgA method (3.16), under the assumption that the weak solution  $u$  of the model problem (2.1) has less regularity, that is  $u \in W_{\mathcal{S}}^{l,p} := W^{1,2}(\Omega) \cap W^{l,p}(\mathcal{S}(\Omega))$ ,  $l \geq 2, p \in (\frac{2d}{d+2(l-1)}, 2]$ . Problems with low regularity solutions can be found in several cases, as for example, when the domain has singular boundary points, points with changing boundary conditions, see e.g. [17], [11], even in particular choices of the discontinuous diffusion coefficient, [21]. We use the enlarged space  $W_h^{l,p} = W_{\mathcal{S}}^{l,p} + \mathbb{B}_h(\mathcal{S}(\Omega))$  and will show that the dG-IgA method converges in optimal rate with respect to  $\|\cdot\|_{dG}$  norm defined in (4.19). We develop our analysis inspired by the techniques used in [34], [27]. A basic tool that we will use is the Sobolev embeddings theorems, see [1],[15]. Let  $l = j + m \geq 2$ , then for  $j = 0$  or  $j = 1$  it holds that

$$(5.1) \quad \|u\|_{W^{j,2}(\Omega_i)} \leq C(l, p, 2, \Omega_i) \|u\|_{W^{l,p}(\Omega_i)}, \text{ for } p > \frac{2d}{d+2m}.$$

We start by proving estimates on how well the quasi-interpolant  $\Pi_h u$  defined in Lemma 4.8 approximates  $u \in W^{l,p}(\Omega_i)$ .

LEMMA 5.1. (*Approximation estimates*). *Let  $u \in W^{l,p}(\Omega_i)$  with  $l \geq 2, p \in (\max\{1, \frac{2d}{d+2(l-1)}\}, 2]$  and let  $E = \Phi_i(\hat{E}), \hat{E} \in T_{h_i, \hat{D}}^{(i)}$ . Then for  $0 \leq m \leq l \leq k+1$ , there exist constants  $C_i := C_i(\max_{l_0 \leq l} \|D^{l_0} \Phi_i\|_{L^\infty(\Omega_i)}), \|u\|_{W^{l,p}(\Omega_i)}$ , such that*

$$(5.2) \quad \sum_{E \in T_{h_i, \Omega_i}^{(i)}} |u - \Pi_h u|_{W^{m,p}(E)}^p \leq h_i^{p(l-m)} C_i.$$

Moreover, we have the following estimates

$$(5.3a) \quad \bullet \ h_i^\beta \|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|_{L^p(F_{ij})}^p \leq C_i C_{d,p} h_i^{p(l-1)-1+\beta},$$

$$(5.3b) \quad \bullet \left( \frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i} \right) \|u - \Pi_h u\|_{L^2(F_{ij})}^2 \leq \\ C_i \alpha^{(j)} \frac{h_i}{h_j} \left( h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)}^p \right)^2 + C_j \alpha^{(i)} \frac{h_j}{h_i} \left( h_j^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_j)} \right)^2 + \\ C_j \left( h_j^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_j)} \right)^2 + C_i \left( h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2,$$

$$(5.3c) \quad \bullet \|u - \Pi_h u\|_{dG}^2 \leq \sum_{i=1}^N C_i \left( h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2 + \\ \sum_{F_{ij} \in \mathcal{F}} C_i \alpha^{(j)} \frac{h_i}{h_j} \left( h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2,$$

where  $\delta(p, d) = l + (\frac{d}{2} - \frac{d}{p} - 1)$ .

*Proof.* We give the proof of (5.2) based on the results of Chap 12 in [32]. Given  $f \in W^{l,p}(\hat{D})$ , there exists a tensor-product polynomial  $T^m f$  of order  $m$ , such that, for every  $\hat{E} \in T_{h_i, \hat{D}}^{(i)}$  the estimate

$$(5.4) \quad |f - T^m f|_{W^{m,p}(\hat{E})} \leq C_{d,l,m} h_i^{l-m} |f|_{W^{l,p}(D_{\hat{E}}^{(i)})},$$



holds, cf. [5] and [32]. Because of  $m \leq k$  holds  $\Pi_h(T^m f) = T^m f$  and  $\|\Pi_h f\|_{L^p(\hat{E})} \leq C\|f\|_{L^p(D_{\hat{E}}^{(i)})}$ . Hence, we have that

$$\begin{aligned}
 (5.5) \quad |u - \Pi_h u|_{W^{m,p}(\hat{E})} &\leq |u - T^m u|_{W^{m,p}(\hat{E})} + |\Pi_h u - T^m u|_{W^{m,p}(\hat{E})} \\
 &\leq |u - T^m u|_{W^{m,p}(\hat{E})} + |\Pi_h(u - T^m u)|_{W^{m,p}(\hat{E})} \\
 &\leq C_1 h_i^{l-m} |u|_{W^{l,p}(D_{\hat{E}}^{(i)})} + C_2 h_i^{-m+\frac{d}{p}-\frac{d}{p}} |\Pi_h(u - T^m u)|_{L^p(\hat{E})} \quad (\text{by (4.10)}) \\
 &\leq C_1 h_i^{l-m} |u|_{W^{l,p}(D_{\hat{E}}^{(i)})} + C_2 h_i^{-m} |u - T^m u|_{L^p(\hat{E})} \quad (\text{by (5.4)}) \\
 &\leq C h_i^{l-m} |u|_{W^{l,p}(D_{\hat{E}}^{(i)})}.
 \end{aligned}$$

Recalling (3.9), the above inequality is expressed on every  $E \in T_{h_i, \Omega_i}^{(i)}$ . Then, taking the  $p - th$  power and summing over the elements we obtain the estimate (5.2).

We consider now the interface  $F_{ij} = \partial\Omega_i \cap \Omega_j$ . Applying (4.9) and using the uniformity of the mesh we get

$$\begin{aligned}
 (5.6) \quad h_i^\beta \|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|_{L^p(F_{ij})}^p &\leq C_i C_{d,p} h_i^\beta \left( \frac{1}{h_i} \|\nabla u^{(i)} - \nabla \Pi_h u^{(i)}\|_{L^p(\Omega_i)}^p + \right. \\
 &\quad \left. h_i^{p-1} \|\nabla^2 u^{(i)} - \nabla^2 \Pi_h u^{(i)}\|_{L^p(\Omega_i)}^p \right) \leq^{\text{by (5.2)}} C_i C_{d,p} h_i^{p(l-1)-1+\beta}.
 \end{aligned}$$

To prove (5.3b), we again make use of the trace inequality (4.9)

$$\begin{aligned}
 (5.7) \quad \frac{\alpha^{(i)}}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 &\leq C_i C_{d,p} \alpha^{(i)} \left( \frac{1}{h_i^2} \int_{\Omega_i} |u^{(i)} - \Pi_h u^{(i)}|^2 dx \right. \\
 &\quad \left. + \int_{\Omega_i} |\nabla(u^{(i)} - \Pi_h u^{(i)})|^2 dx \right) = \\
 &C_i C_{d,p} \alpha^{(i)} \left( \frac{1}{h_i^2} \sum_{E \in T_{h_i, \Omega_i}^{(i)}} \int_E |u^{(i)} - \Pi_h u^{(i)}|^2 dx + \sum_{E \in T_{h_i, \Omega_i}^{(i)}} \int_E |\nabla(u^{(i)} - \Pi_h u^{(i)})|^2 dx \right).
 \end{aligned}$$

The Sobolev embedding (5.1) gives

$$(5.8) \quad \|u\|_{L^2(D_p)} \leq C(p, 2, D_p) (\|u\|_{L^p(D_p)}^p + |u|_{W^{1,p}(D_p)}^p)^{\frac{1}{p}}.$$

Using the scaling arguments, see (4.6), and the bounds (3.9) we can derive the corresponding expression of (5.8) on every  $E \in T_{h_i, \Omega_i}^{(i)}$ ,

$$h_i^{-\frac{d}{2}} \|u\|_{L^2(E)} \leq C_i h_i^{-\frac{d}{p}} (\|u\|_{L^p(E)}^p + h_i^p |u|_{W^{1,p}(E)}^p)^{\frac{1}{p}},$$

where a straight forward computation gives

$$(5.9) \quad h_i^{-2} \|u\|_{L^2(E)}^2 \leq C_i h_i^{2(\frac{d}{2}-\frac{d}{p}-1)} (\|u\|_{L^p(E)}^p + h_i^p |u|_{W^{1,p}(E)}^p)^{\frac{2}{p}}.$$

Proceeding in the same manner, we can show

$$(5.10) \quad \|u\|_{W^{1,2}(E)}^2 \leq C_i h_i^{2(\frac{d}{2}-\frac{d}{p}-1)} (\|u\|_{L^p(E)}^p + h_i^p |u|_{W^{1,p}(E)}^p + h_i^{2p} |u|_{W^{2,p}(E)}^p)^{\frac{2}{p}}.$$

Setting in (5.9) and (5.10)  $u := u^{(i)} - \Pi_h u^{(i)}$  and applying the approximation estimate (5.2), we obtain that

$$\begin{aligned}
 (5.11) \quad & \sum_{E \in T_{h_i, \Omega_i}^{(i)}} \alpha^{(i)} (h_i^{-2} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(E)}^2 + \|u^{(i)} - \Pi_h u^{(i)}\|_{W^{1,2}(E)}^2) \\
 & \leq \sum_{E \in T_{h_i, \Omega_i}^{(i)}} (\alpha^{(i)} C_i h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(D_E^{(i)})})^2 \leq (\text{note that } f(x) = (a^x + b^x)^{\frac{1}{x}} \downarrow) \\
 & \alpha^{(i)} C_i \left( \sum_{E \in T_{h_i, \Omega_i}^{(i)}} (h_i^{lp+p(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(D_E^{(i)})}^p) \right)^{\frac{2}{p}} \leq \alpha^{(i)} C_i \left( h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2.
 \end{aligned}$$

Moreover, by (5.11) we can deduce that

$$(5.12) \quad \frac{\alpha^{(j)} h_i}{h_j} \frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 \leq C_i \frac{\alpha^{(j)} h_i}{h_j} \left( h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2,$$

similarly

$$(5.13) \quad \frac{\alpha^{(i)} h_j}{h_i} \frac{1}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|_{L^2(F_{ji})}^2 \leq C_i \frac{\alpha^{(i)} h_j}{h_i} \left( h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \right)^2.$$

Now, we return to the left hand side of (5.3b) and use (5.11), (5.12) and (5.13), to obtain

$$\begin{aligned}
 (5.14) \quad & \left( \frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i} \right) \|u - \Pi_h u\|_{L^2(F_{ij})}^2 \leq \\
 & \frac{\alpha^{(j)} h_i}{h_j} \frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 + \frac{\alpha^{(i)} h_j}{h_i} \frac{1}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|_{L^2(F_{ji})}^2 \\
 & + \frac{\alpha^{(j)}}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|_{L^2(F_{ji})}^2 + \frac{\alpha^{(i)}}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|_{L^2(F_{ij})}^2 \\
 & \leq C_i \frac{\alpha^{(j)} h_i}{h_j} \left( h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2 + C_j \frac{\alpha^{(i)} h_j}{h_i} \left( h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \right)^2 \\
 & + C_j \left( h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \right)^2 + C_i \left( h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \right)^2.
 \end{aligned}$$

For the proof (5.3c), we recall the definition (4.19) for  $u - \Pi_h u$  and have

$$\begin{aligned}
 (5.15) \quad \|u - \Pi_h u\|_{dG}^2 &= \sum_{i=1}^N \left( \alpha^{(i)} \|\nabla(u^{(i)} - \Pi_h u^{(i)})\|_{L^2(\Omega_i)}^2 \right. \\
 & \quad \left. + \sum_{F_{ij} \in \mathcal{F}} \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|u - \Pi_h u\|_{L^2(F_{ij})}^2 \right).
 \end{aligned}$$

Estimating the first term on the right hand side in (5.15) by (5.2) and the second term by (5.3b), the approximation estimate (5.3c) follows.

□

We need further discrete coercivity, consistency and boundedness. The discrete coercivity (Lemma 4.6) can also be applied here. Using the same arguments as in Lemma 4.9, we can prove the consistency for  $u$ . Due to assumed regularity of the solution, the normal interface flux  $(\alpha \nabla u)|_{\Omega_i} \cdot \mathbf{n}_{F_{ij}}$  belongs (in general) to  $L^p(F_{ij})$ . Thus, we need to prove the boundedness for  $a_h(.,.)$  by estimating the flux terms (3.16d) in different way than this in Lemma 4.7. We work in a similar way as in [28] and show bounds for the interface fluxes in  $\|\cdot\|_{L^p}$  setting.

LEMMA 5.2. *There is a constant  $C := C(p, 2)$  such that the following inequality for  $(u, \phi_h) \in W_h^{l,p} \times \mathbb{B}_h(\mathcal{S}(\Omega))$  holds true*

$$(5.16) \quad \sum_{F_{ij} \in \mathcal{F}} \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket ds \leq$$

$$C \left( \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|_{L^p(F_{ij})}^p + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}} \|\phi_h\|_{dG},$$

where  $\gamma_{p,d} = \frac{1}{2}d(p-2)$ .

*Proof.* For the interface edge  $e_{ij} \subset F_{ij}$  Hölder inequality yield

$$(5.17) \quad \frac{1}{2} \int_{e_{ij}} \frac{1}{2} |\alpha^{(i)} \nabla u^{(i)} + \alpha^{(j)} \nabla u^{(j)}| |\llbracket \phi_h \rrbracket| ds \leq$$

$$C \int_{e_{ij}} (\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(i)}| \frac{\alpha^{(i)\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} |\llbracket \phi_h \rrbracket| + (\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(j)}| \frac{\alpha^{(j)\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} |\llbracket \phi_h \rrbracket| ds$$

$$\leq C (\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(i)}\|_{L^p(e_{ij})} \frac{\alpha^{(i)\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} \|\llbracket \phi_h \rrbracket\|_{L^q(e_{ij})}$$

$$+ C (\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(j)}\|_{L^p(e_{ij})} \frac{\alpha^{(j)\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} \|\llbracket \phi_h \rrbracket\|_{L^q(e_{ij})}.$$

We employ the inverse inequality (4.18) with  $p = q > 2$ ,  $q = 2$  and use the analytical form  $\frac{1+\gamma_{p,d}}{p} = \frac{2+d(p-2)}{2p}$  to express the jump terms in (5.17) in the *convenient  $L^2$  form* as follows

$$(5.18) \quad \frac{\alpha^{(i)\frac{1}{q}}}{h_i^{\frac{2+d(p-2)}{2p}}} \|\llbracket \phi_h \rrbracket\|_{L^q(e_{ij})} \leq C_{inv,p,2} \alpha^{(i)\frac{1}{q}} h_i^{(d-1)(\frac{1}{q}-\frac{1}{2})-\frac{2+d(p-2)}{2p}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})}$$

$$\leq C_{inv,p,2} \alpha^{(i)\frac{1}{q}} h_i^{\frac{-1}{2}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})}.$$

Inserting the result (5.18) into (5.17) and summing over all  $e_{ij} \in F_{ij}$  we obtain for

$q > 2$ ,

$$\begin{aligned}
(5.19) \quad & \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket ds \leq C \sum_{e_{ij} \in F_{ij}} \int_{e_{ij}} |\alpha^{(i)} \nabla u^{(i)} + \alpha^{(j)} \nabla u^{(j)}| \llbracket \phi_h \rrbracket ds \\
& \leq C \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|_{L^p(e_{ij})}^p \right)^{\frac{1}{p}} \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(i)} \left( \frac{1}{h_i^{\frac{1}{2}}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})} \right)^q \right)^{\frac{1}{q}} \\
& \quad + C \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|_{L^p(e_{ij})}^p \right)^{\frac{1}{p}} \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(j)} \left( \frac{1}{h_j^{\frac{1}{2}}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})} \right)^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Now, using that the function  $f(x) = (\lambda \alpha^x + \lambda \beta^x)^{\frac{1}{x}}$ ,  $\lambda > 0, x > 2$  is decreasing, we estimate the “q-power terms” in the sum of the right hand side in (5.19) as follows

$$\begin{aligned}
(5.20) \quad & \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(j)} \left( \frac{1}{h_j^{\frac{1}{2}}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})} \right)^q \right)^{\frac{1}{q}} \leq \left( \sum_{e_{ij} \in F_{ij}} \alpha^{(j)} \left( \frac{1}{h_j^{\frac{1}{2}}} \|\llbracket \phi_h \rrbracket\|_{L^2(e_{ij})} \right)^2 \right)^{\frac{1}{2}} \\
& \leq \left( \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Applying (5.20) into (5.19) we get

$$\begin{aligned}
(5.21) \quad & \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} \llbracket \phi_h \rrbracket ds \leq \\
& 2C \left( \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|_{L^p(F_{ij})}^p + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}} \\
& \quad \left( \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

We sum over all  $F_{ij} \in \mathcal{F}$  in (5.21) and consequently we apply Hölder inequality

$$\begin{aligned}
(5.22) \quad & \frac{1}{2} \sum_{F_{ij} \in \mathcal{F}} \int_{F_{ij}} \{\alpha \nabla u\} \llbracket \phi_h \rrbracket ds \leq \\
& 2C \left( \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|_{L^p(F_{ij})}^p + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}} \\
& \quad \left( \sum_{F_{ij} \in \mathcal{F}} \left( \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.
\end{aligned}$$

Following in much the same arguments as in proof of (5.20), we can bound the second  $\sum_{F_{ij}}$  in (5.22) as

$$\begin{aligned}
(5.23) \quad & \left( \sum_{F_{ij} \in \mathcal{F}} \left( \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \leq \\
& \left( \sum_{F_{ij} \in \mathcal{F}} \left( \frac{\mu \alpha^{(i)}}{h_i} + \frac{\mu \alpha^{(j)}}{h_j} \right) \|\llbracket \phi_h \rrbracket\|_{L^2(F_{ij})}^2 \right)^{\frac{1}{2}} \leq \|\phi_h\|_{dG}.
\end{aligned}$$

Using (5.23) in (5.22), we can easily obtain (5.16).

□

LEMMA 5.3. (*boundedness*) *There is a  $C := C_{p,2}$  independent of  $h_i$  such that  $\forall (u, \phi_h) \in W_h^{l,p} \times \mathbb{B}_h(\mathcal{S}(\Omega))$*

$$(5.24) \quad a_h(u, \phi_h) \leq C(\|u\|_{dG}^p + \sum_{F_{ij} \in \mathcal{F}} h_i^{1+\gamma_{p,d}} \alpha^{(i)} \|\nabla u^{(i)}\|_{L^p(F_{ij})}^p + h_j^{1+\gamma_{p,d}} \alpha^{(j)} \|\nabla u^{(j)}\|_{L^p(F_{ij})}^p)^{\frac{1}{p}} \|\phi_h\|_{dG},$$

*Proof.* We estimate the terms of  $a_h(u, \phi_h)$  in (3.16b) separately. Applying Cauchy-Schwartz for the terms (3.16c) and (3.16e) we have

$$(5.25a) \quad \sum_{i=1}^N a_i(u, \phi_h) \leq C \|u\|_{dG} \|\phi_h\|_{dG}$$

$$(5.25b) \quad \sum_{i=1}^N p_i(u, \phi_h) \leq C \|u\|_{dG} \|\phi_h\|_{dG}.$$

For the term (3.16d) we use Lemma 5.2

$$(5.26) \quad \sum_{i=1}^N s_i(u, \phi_h) \leq C \left( \sum_{F_{ij} \in \mathcal{F}} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|_{L^p(F_{ij})}^p + \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}} \|\phi_h\|_{dG},$$

Combining (5.25) with (5.26) we can derive (5.24).

□

Next, we prove the main convergence result of this section.

THEOREM 5.4. *Let  $u \in W_S^{l,p}$ ,  $l \geq 2$ ,  $p \in (\max\{1, \frac{2d}{d+2(l-1)}\}, 2]$  be the solution of (2.2a). Let  $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$  be the dG-IgA solution of (3.16a) and  $\Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$  is the interpolant of Lemma 5.1. Then there are  $C_i := C_i(\max_{l_0 \leq l} \|D^{l_0} \Phi_i\|_{L^\infty(E)}, \|u\|_{W^{l,p}(\Omega_i)})$ , such that*

$$(5.27) \quad \|u - u_h\|_{dG} \leq \sum_{i=1}^N \left( C_i \left( h_i^{\delta(p,d)} + \sum_{F_{ij} \in \mathcal{F}} \alpha^{(j)} \frac{h_i}{h_j} h_i^{\delta(p,d)} \right) \|u\|_{W^{l,p}(\Omega_i)} \right),$$

where  $\delta(p, d) = l + (\frac{d}{2} - \frac{d}{p} - 1)$ .

*Proof.* Since  $(u_h - \Pi_h u) \in \mathbb{B}_h(\mathcal{S}(\Omega))$  by the discrete coercivity (4.21) we have

$$(5.28) \quad \|u_h - \Pi_h u\|_{dG}^2 \leq a_h(u_h - \Pi_h u, u_h - \Pi_h u).$$

By orthogonality we have

$$\begin{aligned}
\|u_h - \Pi_h u\|_{dG}^2 &\leq a_h(u_h - \Pi_h u, u_h - \Pi_h u) = \\
&a_h((u_h - u) + (u - \Pi_h u), u_h - \Pi_h u) = a_h(u - \Pi_h u, u_h - \Pi_h u) \\
&\leq C \left( \|u - \Pi_h u\|_{dG} + \left( \sum_{F_{ij} \in \mathcal{F}} h_i^{1+\gamma_{p,d}} \alpha^{(i)} \|\nabla u^{(i)} - \Pi_h u^{(i)}\|_{L^p(F_{ij})}^p \right. \right. \\
&\quad \left. \left. + h_j^{1+\gamma_{p,d}} \alpha^{(j)} \|\nabla u^{(j)} - \Pi_h u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}} \right) \|u_h - \Pi_h u\|_{dG},
\end{aligned}$$

where immediately we get

$$\begin{aligned}
(5.29) \quad \|u_h - \Pi_h u\|_{dG} &\leq \|u - \Pi_h u\|_{dG} + \left( \sum_{F_{ij} \in \mathcal{F}} h_i^{1+\gamma_{p,d}} \alpha^{(i)} \|\nabla u^{(i)} - \Pi_h u^{(i)}\|_{L^p(F_{ij})}^p \right. \\
&\quad \left. + h_j^{1+\gamma_{p,d}} \alpha^{(j)} \|\nabla u^{(j)} - \Pi_h u^{(j)}\|_{L^p(F_{ij})}^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Now, using triangle inequality, the approximation estimates (5.3) and the bound (5.16) in (5.29), we obtain

$$\begin{aligned}
(5.30) \quad \|u_h - u\|_{dG} &\leq \|u_h - \Pi_h u\|_{dG} + \|u - \Pi_h u\|_{dG} \leq \\
&\sum_{i=1}^N C_i h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)} + \sum_{F_{ij} \in \mathcal{F}} C_i \frac{\alpha^{(j)} h_i}{h_j} h_i^{\delta(p,d)} \|u\|_{W^{l,p}(\Omega_i)},
\end{aligned}$$

which is the required error estimate (5.27).

□

**6. Numerical examples.** In this section, we present a series of numerical examples to validate numerically the theoretical results, which were previously shown. We restrict ourselves for a model problem in  $\Omega = (\frac{-1}{2}, \frac{1}{2})^{d=3}$ , with  $\Gamma_D = \partial\Omega$ . The domain  $\Omega$  is subdivided in four equal sub-domains  $\Omega_i, i = 1, \dots, 4$ , where for simplicity every  $\Omega_i$  is initially partitioned into a mesh  $T_{h_i, \Omega_i}^{(i)}$  with  $h := h_i = h_j, i \neq j, j = 1, \dots, 4$ . Successive uniform refinements are performed on every  $T_{h_i, \Omega_i}^{(i)}$  in order to compute numerically the convergence rates. We set the diffusion coefficient equal to one.

All the numerical tests have been performed in G+SMO<sup>1</sup>, which is a generic object oriented C++ library for IgA computations. For the reasons mentioned in Remark 3.1, the practical implementation in G+SMO is based on SIP dG-IgA. In the first test, the data  $u_D$  and  $f$  in (2.1) are determined so that the exact solution is given by  $u(x) = \sin(2.5\pi x) \sin(2.5\pi y) \sin(2.5\pi z)$  (smooth test case). The first two columns of Table 1 display the convergence rates. As it was expected, the convergence rates are optimal. In the second case, the exact solution is  $u(x) = |x|^\lambda$ . The parameter  $\lambda$  is chosen such that  $u \in W^{l,p=1.4}(\Omega)$ . In the last columns of Table 1, we display the convergence rates for degree  $k = 2, k = 3$  and  $l = 2, l = 3$ . We observe that, for each of the two different tests, the error in the dG-norm behaves according to the main error estimate given by (5.27).

REMARK 6.1. *In a forthcoming paper, we will present graded mesh techniques in dG-IgA methods for treating problems with low regularity solutions. We will show,*

<sup>1</sup>G+SMO: <http://www.gs.jku.at/gs-gismo.shtml>

	highly smooth		$k = 2$		$k = 3$	
$\frac{h}{2^s}$	$k = 2$	$k = 3$	$l = 2$	$l = 3$	$l = 2$	$l = 3$
-	Convergence rates					
$s = 0$	-	-	-	-	-	-
$s = 1$	0.15	2.91	0.62	0.76	0.24	1.64
$s = 2$	2.34	2.42	0.29	1.10	0.28	0.89
$s = 3$	2.08	3.14	0.35	1.32	0.47	1.25
$s = 4$	2.02	3.04	0.35	1.36	0.36	1.37

TABLE 1

The numerical convergence rates of the dG-IgA method.

how to construct graded refined mesh in the vicinity of the singular points of  $u$ , in order to get the optimal approximation order as in the case of having smooth  $u$ .

**7. Conclusions.** In this paper, we presented theoretical error estimates of the dG-IgA method applied to a model elliptic problem with discontinuous coefficients. The problem was discretized according to IgA methodology using discontinuous  $\mathbb{B}$ -Spline spaces. Due to global discontinuity of the approximate solution on the sub-domain interfaces, dG discretizations techniques were utilized. In the first part, we assumed higher regularity for the exact solution, that is  $u \in W^{l \geq 2, 2}$ , and we showed optimal error estimates with respect to  $\|\cdot\|_{dG}$ . In the second part, we assumed low regularity for the exact solution, that is  $u \in W^{l \geq 2, p \in (\frac{2d}{d+2(l-1)}, 2)}$ , and applying the Sobolev embedding theorem we proved optimal convergence rates with respect to  $\|\cdot\|_{dG}$ . The theoretical error estimates were validated by numerical tests. The results can obviously be carried over to diffusion problems on open and closed surfaces as studied in [25], and to more general second-order boundary value problems like linear elasticity problems as studied in [2].

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